Mathematics 552 **Algebra II** **Spring 2017**

Week 6 Semisimple rings, the Artin-Wedderburn Theorem and Group Representations
Jacobson II: 5.1, 5.2, 5.3

1. Let V be an infinite dimensional vector space over a division ring D with countable basis $e_1, e_2, \ldots$. Let $R = End_D(V)$ be the ring of F-linear maps of V to itself, with product the composition of maps.

   a) Show that the set $I \subset R$ of all linear maps with finite dimensional image is a two-sided ideal in R.

   b) Show that the only two sided ideals of R are $0, R, I$. Show that I is a maximal two-sided ideal in R and that the ring R/I is a simple ring.

   c) Show that the set I_j consisting of all endomorphisms vanishing on e_n when 2^j divides n is a left ideal of R. Show that the left ideals $I + I_j$ are distinct for distinct j and form an increasing chain of left ideals and that R/I is a simple ring which is not semisimple.

2. Let K be a field, let G be a group and let $K[G]$ be the group ring of G.

   a) Show that if kernel of the map $K[G] \to K$ which sends $\sum a_g g \in K[G]$ to $\sum a_g$ has a complement that it is a one dimensional K-vector space which is an eigenspace for the linear transformation multiplication by g on $K[G]$.

   b Show that when the characteristic of K is a prime number p and $K[G]$ is a semisimple ring then G has no elements of order p.

   c) Show that the converse of Maschke's theorem holds: The group ring $K[G]$ of a finite group G is semisimple if and only if the order of G is not divisible by the characteristic of K.

3. Show that if R is a semisimple ring and M is a finitely generated left R-module, then $End_R(M)$ is a semisimple ring.

4. Jacobson II 5.3.1

5. Jacobson II 5.3.3 (Take G to be a finite group and let F be a finite dimensional vector space over K so that Krull-Schmidt can perhaps be utilized. It is not clear how general Jacobson was considering the group and degree of field extension to be. See if you can find an alternate proof covering general groups G and any extensions K/F of an infinite field F.)