

RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination

Session on Algebra

The Qualifying Examination consists of three two-hour sessions. This is the session on Algebra. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
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Part I. Answer all questions.

1. Let G be a finite group and let p be the smallest prime divisor of $|G|$. Show that any subgroup H of G of index p is normal in G .

Hint: Consider maps $G \rightarrow S_p$.

2. Let R be a principal ideal domain, and F a free R -module of finite rank. Show that any surjective R -module homomorphism $f : F \rightarrow F$ is an isomorphism.

3. Let R be a noetherian domain with the property that if I and J are principal ideals in R , then $I + J$ is also a principal ideal. Prove that R is a principal ideal domain.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let X, Y be nonzero 3×3 matrices over the real numbers $\mathbb{R}$ satisfying

$$X^3 + X = 0.$$

a) Show that X and Y need not be similar over the complex numbers $\mathbb{C}$.

b) Show that X and Y must be similar over $\mathbb{R}$.

5. a) Show that every finite subgroup of $\mathbb{C}^\times$ is cyclic.

b) Suppose that A is a finite abelian group, and $f : A \rightarrow \mathbb{C}^\times$ is a homomorphism with $f(A) \neq \{1\}$. Show that $\sum_{a \in A} f(a) = 0$ in $\mathbb{C}$.

End of Session on Algebra

RUTGERS UNIVERSITY
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Written Qualifying Examination

Session on Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the session on Complex Variables and Advanced Calculus. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

Problem 1: Evaluate

$$\int_0^\pi \tan(t + ai) dt$$

for $a \neq 0$.

Problem 2: Let

$$F(z) = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n}.$$

Find its power series $\sum_{k=1}^{\infty} a_k z^k$ and find its radius of convergence.

Problem 3: Find the number of roots of

$$z^4 - 6z + 3 = 0$$

in $D = \{1 < |z| < 2\}$.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

Problem 4: Let $c > 0$ and

$$D = \{|z| > 1, |z - c| < 1\}, \quad F(z) = \frac{z - z_1}{z - z_2},$$

where $z_1, z_2 \in \mathbb{C}$ are the intersection points of the circles $|z| > 1$ and $|z - c| = 1$, with $\operatorname{Im} z_1 < 0$ and $\operatorname{Im} z_2 > 0$. Find the value of c such that $F(D)$ is bounded by two rays with angle equal to $\frac{\pi}{3}$. Then find $F(D)$.

Problem 5: Let

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^8} z^{3^n},$$

which has convergence radius 1. (Thus $f(z)$ is a well defined holomorphic function over the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$.)

1. Prove that $f(z)$ does not admit a holomorphic extension to a neighborhood of 1 in $\mathbb{C}$. Namely, there do not exist a neighborhood U of 1 in the complex plane $\mathbb{C}$ and a holomorphic function g over U such that $f|_{U \cap \Delta} = g|_{U \cap \Delta}$.
2. Further show that the unit disk is the natural defining domain of $f(z)$. Namely, there do not exist a domain Ω strictly larger than the unit disk and a holomorphic function F defined over Ω such that the restriction of F to the unit disk is $f(z)$.

End of Session on Complex Variables and Advanced Calculus

RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination
January 2024

Session 1. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
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- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let $\tau_h f(x) = f(x - h)$ be the translation. Find all $p \in [1, \infty]$ for which $f \in L^p(\mathbb{R})$ implies that $\lim_{h \rightarrow 0} \|f - \tau_h f\|_{L^p(\mathbb{R})} = 0$ (and justify your answer).
2. Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be a sequence of measurable functions with $|f_n(x)| \leq 1$ for a.e. x . Let

$$g_n(x) = \int_0^x f_n(t) dt.$$

Show that there exists an absolutely continuous g and a subsequence $n_k \rightarrow \infty$ such that $g_{n_k} \rightarrow g$ in $C([0, 1])$.

3. Let $f_n, g_n \in L^2([0, 1])$ be sequences. Assume that for $f \in L^2([0, 1])$ and $g : [0, 1] \rightarrow \mathbb{R}$ measurable, $f_n \rightarrow f$ in $L^2([0, 1])$ norm while $g_n \rightarrow g$ almost everywhere. Also assume that $\|g_n\|_{L^2([0, 1])} \leq 1$. Show that the product $f g$ is integrable, and that

$$\int f_n g_n \rightarrow \int f g.$$

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let (X, d) be a metric space, and let q be another metric on X . Assume that, for any $x \in X$, a sequence $\{x_n\} \subset X$ converges to x with respect to d if and only if it converges to x with respect to q .
 - (a) Show that d and q induce the same topology on X .
 - (b) Give an example showing that d and q need not be equivalent (*equivalent* here means that there is a constant $C > 0$ such that $C^{-1}d(x, y) \leq q(x, y) \leq Cd(x, y)$ for all $x, y \in X$).
5. Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be a sequence of Lebesgue integrable functions with $f_n \rightarrow f$ in $L^1([0, 1])$.

(a) Define, for any $t \in \mathbb{R}$, the function

$$g(x) = \begin{cases} 1 & x > t \\ 0 & x \leq t. \end{cases}$$

Show that there exists a subsequence n_k such that

$$g \circ f \leq \liminf_{k \rightarrow \infty} g \circ f_{n_k}$$

almost everywhere.

(b) Show that, for any $t \in \mathbb{R}$,

$$|\{x \in [0, 1] : f(x) > t\}| \leq \liminf_{n \rightarrow \infty} |\{x \in [0, 1] : f_n(x) > t\}|.$$

(c) Show that for all but countably many t , $\lim_{n \rightarrow \infty} |\{x \in [0, 1] : f(x) > t\}|$ exists and equals $|\{x \in [0, 1] : f(x) > t\}|$.