

RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination

August 2021

Session on Algebra

The Qualifying Examination consists of three two-hour sessions. This is the session on Algebra. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
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Part I. Answer all questions.

1. Let G be a group and $Z(G)$ the center of G . Show that the group $G/Z(G)$ does not have prime order. Find a group G such that $G/Z(G)$ has 4 elements.

2. Show that every prime ideal P in $\mathbf{Z}[x]$ which is not principal contains a prime number

3. Show that every finite noncyclic group is a finite union of proper subgroups, and that if a group maps surjectively to a finite noncyclic group then it is a finite union of proper subgroups and use this to determine for which positive integers the product of n copies of the integers is a finite union of proper subgroups.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let A and B be two square matrices over a field F . Suppose $\text{diag}(A, A)$ and $\text{diag}(B, B)$ are similar. Show that A and B are similar.

5.

- (A) Suppose that p and q are distinct primes and a group G is generated by elements of order p and also by elements of order q . Show that any homomorphism of G to an abelian group is trivial.
- (B) Show that for $n \geq 5$ the alternating group A_n of even permutations of n objects is generated by elements of order 2, and also by elements of order 3, so that for such n the only homomorphisms to abelian groups are trivial.

End of Session on Algebra

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Session on Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the session on Complex Variables and Advanced Calculus. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

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Part I. Answer all questions.

- 1.** $0 < R < \pi/2$ is fixed. Prove that for all sufficiently large $n \in \mathbb{N}$ the polynomial

$$P_n(z) := 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!}$$

has no roots with absolute value less than R .

- 2.** Use the calculus of residues to explicitly compute $\int_0^\infty \frac{x^n}{1+x^{2n}} dx$. Here $n \geq 2$ is a positive integer.

- 3.** Let $D_0 := \{z \in \mathbb{C} : 0 < |z| < 1\}$. $f : D_0 \rightarrow \mathbb{C}$ is holomorphic on D_0 and satisfies $|f(z)| \leq \log(1/|z|)$ for any $z \in D_0$. Prove that $f \equiv 0$ on D_0 .

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let $U := \{x + iy : -\pi/2 < y < \pi/2\}$.

1. Give an example of an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that $f|_U$ is unbounded, but $|f(z)| = 1$ for any $z \in \partial U$.
2. Construct a sequence of entire functions $\{h_n\}_{n \in \mathbb{N}} : \mathbb{C} \rightarrow \mathbb{C}$ such that
 - $\limsup_{z \in U, z \rightarrow \infty} |h_n(z)| \cdot e^{|\operatorname{Re} z|/3} = 0$ for each $n \in \mathbb{N}$;
 - $\lim_{n \rightarrow \infty} h_n(z) = 1$ for each $z \in \mathbb{C}$
3. Suppose $f : U \rightarrow \mathbb{C}$ is holomorphic, satisfies $|f(z)| \leq e^{|\operatorname{Re} z|/3}$ for any $z \in U$, and continuously extends to $\bar{U}$ with bounded $f|_{\partial U}$. Prove that f must be bounded. **Hint:** Look at functions $f(z)h_n(z)$ as $n \rightarrow \infty$.

5. Let $D := \{z \in \mathbb{C} : |z| < 1\}$. $f : D \rightarrow \mathbb{C}$ is holomorphic, injective and satisfies $f'(0) = 1$. Prove that the area of $f(D)$ is at least π .

End of Session on Complex Variables and Advanced Calculus

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Session on Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the session on Real Variables and Elementary Point-Set Topology. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

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Part I. Answer all questions.

1. (a) For a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, write down the definition of

$$\limsup_{n \rightarrow \infty} a_n .$$

- (b) Prove that, for any sequence of Lebesgue measurable functions $f_n : \mathbb{R} \rightarrow [0, 1]$, $n = 1, 2, \dots$, the function

$$f = \limsup_{n \rightarrow \infty} f_n$$

is also Lebesgue measurable.

2. Consider the ℓ^1 space

$$\ell^1 = \{x = (x_1, x_2, \dots) : x_k \in \mathbb{R}, \|x\|_1 < \infty\}, \quad \|x\|_1 = \sum_{k=1}^{\infty} |x_k| ,$$

and endow the space with the metric $\rho(x, y) = \|x - y\|_1$. Prove that

$$K = \{x = (x_1, x_2, \dots) : |x_k| \leq k^{-2}, k = 1, 2, \dots\}$$

is a compact set in ℓ^1 .

3. We say a function $g : [0, 1] \rightarrow \mathbb{R}$ is Lipschitz if

$$\sup \left\{ \frac{|g(x) - g(y)|}{|x - y|} : x, y \in [0, 1], x \neq y \right\} < \infty .$$

Prove that, for any non-decreasing and absolutely continuous function $f : [0, 1] \rightarrow \mathbb{R}$, there exists a sequence of functions $f_n : [0, 1] \rightarrow \mathbb{R}$, $n = 1, 2, \dots$, such that each f_n is non-decreasing, Lipschitz, and

$$f_1 \leq f_2 \leq \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} f_n = f \quad \text{everywhere on } [0, 1] .$$

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let $f_n : [0, \infty) \rightarrow \mathbb{R}$, $n = 1, 2, \dots$, be such that

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(x) &= e^{-x} \text{ for every } x \in [0, \infty), \\ \sup_n \int_0^\infty |f_n(x)|^2 e^{-x} dx &< \infty, \\ \limsup_{n \rightarrow \infty} \int_0^\infty |f_n(x)| dx &\leq 1. \end{aligned}$$

Prove that

$$\lim_{n \rightarrow \infty} \int_0^\infty |f_n(x) - e^{-x}| dx = 0.$$

5. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \rightarrow [0, 1]$ be $\mathcal{A}$ -measurable.

- (a) Let $\mathcal{B}_{[0,1]}$ denote the Borel σ algebra on $[0, 1]$. Prove that the set

$$H = \{(x, y) : x \in \Omega, y \in [0, f(x)]\} \subset \Omega \times [0, 1]$$

is $(\mathcal{A} \times \mathcal{B}_{[0,1]})$ -measurable.

- (b) Assume further that $\mu(\Omega) < \infty$, and let m denote the Lebesgue measure on $[0, 1]$. Prove that

$$(\mu \times m)(H) = \int_\Omega f d\mu.$$

End of Session on Real Variables and Elementary Point-Set Topology